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The (1+1) Dimensional Dirac Equation With Pseudoscalar Potentials: Path Integral Treatment

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The supersymetric path integrals in solving the problem of relativistic spinning particle interacting with pseudoscalar potentials is examined. The relative propagator is presented by means of path integral, where the spin degrees of freedom are described by odd Grassmannian variables and the gauge invariant part of the effective action has a form similar to the standard pseudoclassical action given by Berezin and Marinov. After integrating over fermionic variables (Grassmannian variables), the problem is reduced to a nonrelativistic one with an effective supersymetric potential. Some explicit examples are considered, where we have extracted the energy spectrum of the electron and the wave functions.

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1. INTRODUCTION

In relativistic quantum mechanics the Klein Gordon and Dirac equations can be considered as first approximation of the field theory when the corrections are perceptible only in the presence of strong fields (Gross, 1993; Bjorken and Drell, 1965; Greiner, 1990). What explains the increased interest of these equations and particularly the importance to find their exact and analytic solutions preferably handier than what exists (Bagrov and Gitman, 1990). It is obvious that such solutions allow a better description and analysis of certain physical phenomena. However, in order to satisfy this deep need of comprehension, the quantum mechanics was reformulated according to other approaches such as the path integral (Feynman and Hibbs, 1965) that remains currently a useful quantization method.

For the Dirac equation which is a fundamental equation in physics, the path integral formulation has n't known the same development, mainly because of the difficulty of inserting the anticommuting γ-matrices by means of paths. However, a

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successful formulation for relativistic spinning particles was elaborated by Fradkin and Gitman 1991 according to the Feynman standard form

$$\int D\,(\text{path})\exp i\,S(\text{path}), \tag{1}$$

where S is a supersymetric action which describes at the same time the external motion and internal one related to the spin of the particle. Elsewhere, the same problem is reconsidered following the so-called global and local representations by Alexandrou *et al.*, 1998.

Recently, the problem of a relativistic spinning particle interacting with vector and scalar potentials and pseudoscalar potential has been widely discussed (Villalba, 1995, 1997) (see also (Alhaidari *et al.*, 2006) and references therein.). The authors have written the $(1+1)$ dimensional Dirac equation in the more general form (de Castro, 2003; Sinha and Roy, 2005)

$$i\frac{\partial}{\partial t}\psi = \left[\alpha P + \beta m + \mathbb{V}\,(x)\right]\psi, \tag{2}$$

where the potential $\mathbb{V}\,(x)$ has the following Lorentz structure

$$\mathbb{V}\,(x) = V_0\,(x) + \alpha V_1\,(x) + \beta V_s\,(x) + \beta\gamma^5 V_p\,(x)\,. \tag{3}$$

The pieces $V_0\,(x)$ and $V_1\,(x)$ are the two components of Lorentz 2-vector and $V_s\,(x)$, $V_p\,(x)$ stand respectively for the scalar potential and the pseudoscalar one.

For the vector and the scalar potentials the path integral formulation and the pseudoclassical description have been done straightforwardly (Fradkin and Gitman, 1991; Alexandrou *et al.*, 1998). However, to our knowledge, there is no path integral discussion for the pseudoscalar potentials. We mention that the path integral formulation is derived for the problem of Dirac equation with torsion field that has some resemblance to but different from the pseudoscalar potential case (Geyer *et al.*, 2000). The torsion field, which carries the γ^5 matrix, has 4 components and is not a Lorentz pseudoscalar.

Our purpose in this paper is to examine the method of supersymetric path integrals in studying the relativistic spinning particle subjected to a one dimensional pseudoscalar potential. So, we shall treat analytically by the Feynman path integrals the problem of a Dirac particle in position-dependent pseudoscalar interaction.

In the first stage we give a path integral formulation for the propagator that we suggest to calculate according to the global projection and we describe the spin degrees of freedom by odd Grassmannian variables. In second way, we show, after integrating over odd trajectories, that the relative Green's function can be expressed only through bosonic path integrals. The problem will be then reduced to the propagator of Schrödinger particle in an effective supersymetric potential. Finally, we consider some particular examples.

2. FORMULATION OF THE PROBLEM

The path integral formulation elaborated by Fradkin and Gitman 1991 to determine the Green's function solution of

$$[\gamma^\mu(P_\mu - eA_\mu) - m]S^c(x_b, x_a) = -\delta(x_b - x_a) \tag{4}$$

consists in making the operator $[\gamma^\mu(P_\mu - eA_\mu) - m]$ homogeneous in γ-matrices by multiplying it by $\gamma^5(\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3)$ and defining new $\tilde{\gamma}$-matrices; $\tilde{\gamma}^\mu = \gamma^5\gamma^\mu$, $\tilde{\gamma}^5 = \gamma^5$, which obey $[\tilde{\gamma}^m, \tilde{\gamma}^n]_+ = \eta^{mn}$, with $\eta^{mn} = \mathrm{diag}(1, -1, -1, -1, -1)$ and $m, n = \overline{0, 3}, 5$. Since these matrices are considered as Fermi-type operators, the homogeneous operator $\tilde{F} \equiv [\tilde{\gamma}^\mu(P_\mu - eA_\mu) - \tilde{\gamma}^5 m]$ is a pure Fermi operator and its square $\tilde{F}^2$ is a Bose-type operator that can be represented by Schwinger proper time representation. Therefore, the propagator takes an exponential form

$$S^c(x_b, x_a)\gamma^5 = \int d\lambda \int \langle x_b| \exp i[\lambda \tilde{F}^2 + \chi \tilde{F}]|x_a\rangle d\chi, \tag{5}$$

where χ is an odd Grassmannian variable anticommuting with $\tilde{\gamma}$-matrices. This representation leads to a supersymetric action path integral form (Berezin and Marinov, 1975, 1977; Brink *et al.*, 1976, 1977).

Besides, we notice that this representation is generalized to the case of arbitrary dimensions in Gitman (1997) and it is shown that in odd dimensions $(2d + 1)$ the matrix γ^5 does not exist and consequently the procedure of making Dirac operator homogeneous in γ-matrices does not work. However, the path integral formulation is derived rigorously by using other technic.

For the case of the pseudoscalar interaction in $(1 + 1)$ dimension, there exists γ^5, but we cannot find homogeneous operator. It is then difficult to build a local path integral representation. So, we construct a global representation starting from the causal Green's function $S^c(x_b, x_a)$ solution of the equation

$$[\gamma^\mu P_\mu - \gamma^5 V_p(x^1) - m]S^c(x_b, x_a) = -\delta^2(x_b - x_a), \tag{6}$$

where $x \equiv (x^0, x^1)$ and x^1 is the position coordinate. The γ-matrices are given, in $(1 + 1)$ dimension, in terms of Pauli matrices

$$\gamma^0 = \sigma_z, \quad \gamma^1 = i\sigma_y, \quad \gamma^5 = i\gamma^0\gamma^1 = i\sigma_x. \tag{7}$$

Then, we present $S^c(x_b, x_a)$ as a matrix element of an operator $\mathbb{S}^c$

$$S^c(x_b, x_a) = \langle x_b|\mathbb{S}^c|x_a\rangle, \tag{8}$$

where

$$\mathbb{S}^c = \frac{1}{K_-} = K_+ \frac{1}{K_- K_+} \tag{9}$$

and the operators K_- and K_+ are given by

$$K_\pm = [\gamma^\mu P_\mu - \gamma^5 V_p(x^1) \pm m]. \tag{10}$$

This procedure is used in Alexandrou *et al.* (1998) to derive path integral representation for the propagator systematically without the usual five-dimensional extension (i.e. without γ^5) and it is employed also in Gitman (1997) in the case of odd dimensions where there is no γ^5 matrix. However, in the present case, despite of the existing of γ^5, we must use this procedure to obtain a Bose-type operator that has a quadratic form with respect to γ-matrices.

The product K_-K_+ is then

$$K_-K_+ = P^2 - m^2 - V_p^2(x^1) + i\frac{1}{2}F_{mn}\gamma^m\gamma^n \tag{11}$$

where the antisymmetric tensor F_{mn}, that has to be understood as a matrix with lines marked by the first contravariant indices and with columns marked by the second covariant indices, is given by

$$F_{mn} = f_{mn}\frac{\partial}{\partial x^1}V_p(x^1), \tag{12}$$

with

$$f = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{13}$$

In this connection, we notice that for the general case, where $V_p(x)$ is position- and time-dependent potential, the antisymmetric tensor F_{mn} will be given by

$$\begin{aligned} F_{51} &= -F_{15} = \frac{\partial}{\partial x^1}V_p(x), \\ F_{50} &= -F_{05} = \frac{\partial}{\partial x^0}V_p(x), \\ F_{01} &= F_{10} = 0, \end{aligned} \tag{14}$$

however, as it is mentioned in the introduction we interest only to the significant case of the space-dependent potential.

Now, in order to build a global representation we use the relation $\int dx^2|x\rangle\langle x| = 1$. We get

$$S^c(x_b, x_a) = \left[i\gamma^\mu\frac{\partial}{\partial x_b^\mu} - \gamma^5 V_p\left(x_b^1\right) + m\right]G^c(x_b, x_a). \tag{15}$$

Here the operator $[i\gamma^\mu\partial_\mu - \gamma^5 V_p(x^1) + m]$ will eliminate the superfluous states caused by the product K_-K_+ in (9) and the Green's function $G^c(x_b, x_a)$, that we suggest to calculate via path integration, has the following proper time representation

$$G^c(x_b, x_a) = i \int d\lambda \langle x_b | \exp(-i\mathcal{H}(\lambda)) | x_a \rangle, \tag{16}$$

where

$$\mathcal{H}(\lambda) = \lambda \left(-P^2 + m^2 + V_p^2(x^1) - \frac{i}{2} F_{mn} \gamma^m \gamma^n \right). \tag{17}$$

To present $G^c(x_b, x_a)$ by means of a path integral we write, in the beginning, $\exp(-i\mathcal{H}(\lambda)) = [\exp(-i\mathcal{H}(\lambda)\varepsilon)]^N$, with $\varepsilon = 1/N$, and we insert $(N-1)$ identities $\int |x\rangle\langle x| dx = 1$ between all the operators $\exp(-i\varepsilon\mathcal{H}(\lambda))$. Next, we introduce N integrations $\int d\lambda_k \delta(\lambda_k - \lambda_{k-1}) = 1$. We then obtain

$$\begin{aligned} G^c &= i \lim_{\substack{N\to\infty \\ \varepsilon\to 0}} \int d\lambda_0 \int dx_1 dx_2 \ldots dx_{N-1} \int d\lambda_1 d\lambda_2 \ldots d\lambda_N \\ &\times \prod_{k=1}^{N} \langle x_k | \exp(-i\varepsilon\mathcal{H}(\lambda_k)) | x_{k-1} \rangle \, \delta(\lambda_k - \lambda_{k-1}). \end{aligned} \tag{18}$$

As ε is small, we can write

$$\langle x_k | \exp(-i\varepsilon\mathcal{H}(\lambda_k)) | x_{k-1} \rangle \approx \langle x_k | 1 - i\varepsilon\mathcal{H}(\lambda_k) | x_{k-1} \rangle \tag{19}$$

and, as $\mathcal{H}(\lambda)$ has no product of the operators X, P, using the relation $\int |p_k\rangle \langle p_k| \, dp_k = 1$ and taking into account that

$$\langle x_k \mid p_{k'} \rangle = \frac{1}{2\pi} e^{i p_{k'} x_k}, \tag{20}$$

the matrix element (19) can be expressed in the middle point $\tilde{x}_k = (x_k + x_{k-1})/2$

$$\int \frac{dp_k}{(2\pi)^2} \exp\left\{ i \left[p_k \frac{x_k - x_{k-1}}{\varepsilon} - \mathcal{H}(\lambda_k, \tilde{x}_k, p_k) \right] \varepsilon \right\}. \tag{21}$$

The multipliers in (18) are noncommutative due to the γ-matrices structure so that we attribute formally the index k, to γ-matrices, and we introduce the $\mathbb{T}$-product which acts on γ-matrices. Then, using the integral representation for the δ-functions

$$\delta(\lambda_k - \lambda_{k-1}) = \frac{i}{2\pi} \int e^{i\pi_k(\lambda_k - \lambda_{k-1})} d\pi_k, \tag{22}$$

it becomes possible to gather all the multipliers, entering in (18), in one exponent and the Green's function G^c can be expressed as follows

$$G^c = \mathbb{T}\int_0^\infty d\lambda_0 \int Dx \int Dp \int D\lambda \int D\pi \\ \times \exp\left\{ i\int_0^1 d\tau \left[\lambda\left(p^2 - m^2 - V_p^2(x^1)\right) + p\dot{x} + \pi\dot{\lambda} \\ + \lambda \frac{i}{2} F_{mn}\gamma^m\gamma^n\right]\right\}. \tag{23}$$

In order to insert the γ-matrices by means of path integrals we introduce an odd source ρ^μ. We obtain

$$G^c = \mathbb{T}\int_0^\infty d\lambda_0 \int Dx \int Dp \int D\lambda \int D\pi \\ \times \exp\left\{ i\int_0^1 d\tau \left[\lambda\left(p^2 - m^2 - V_p^2(x^1)\right) + p\dot{x} + \pi\dot{\lambda} \\ + \lambda \frac{i}{2} F_{mn}\frac{\delta_\ell}{\delta\rho^m}\frac{\delta_\ell}{\delta\rho^n}\right]\right\} \mathbb{T}\exp\int_0^1 \rho(\tau)\gamma d\tau\Big|_{\rho=0}. \tag{24}$$

Next, we present the quantity $\mathbb{T}\exp\int_0^1 \rho(\tau)\gamma d\tau$ via a path integral over grassmannian odd trajectories (Fradkin and Gitman, 1991; Alexandrou *et al.*, 1998)

$$\mathbb{T}\exp\int_0^1 \rho(\tau)\gamma d\tau = \exp\left(i\gamma^n\frac{\partial_l}{\partial\theta^n}\right)\int_{\psi(0)+\psi(1)=\theta} \mathcal{D}\psi \\ \times \exp\left\{\int_0^1 d\tau[\psi_n\dot{\psi}^n - 2i\,\rho_n\psi^n] + \psi_n(1)\psi^n(0)\right\}, \tag{25}$$

where the measure $\mathcal{D}\psi$ is given by

$$\mathcal{D}\psi = D\psi\left[\int_{\psi(0)+\psi(1)=0} D\psi \exp\left\{\int_0^1 \psi_n\dot{\psi}^n d\tau\right\}\right]^{-1} \tag{26}$$

and θ^n and ψ^n are odd variables, anticommuting with γ-matrices.

Finally, the Green's function G^c will be presented in the Hamiltonian path integral representation as follows:

$$G^c = \exp\left(i\gamma^n\frac{\partial_l}{\partial\theta^n}\right)\int_0^\infty d\lambda_0 \int Dx \int Dp \int D\lambda \int D\pi \int_{\psi(0)+\psi(1)=\theta} \mathcal{D}\psi \\ \times \exp\left\{ i\int_0^1 d\tau \left[\lambda\left(p^2 - m^2 - V_p^2(x^1) + 2iF_{mn}\psi^m\psi^n\right) \\ - i\psi_n\dot{\psi}^n + p\dot{x} + \pi\dot{\lambda}\right] + \psi_n(1)\psi^n(0)\right\}\Big|_{\theta=0}. \tag{27}$$

We notice that, integrating over momenta and separating the gauge-fixing term $\pi\dot{\lambda}$ and the boundary term $\psi_n(1)\psi^n(0)$ we obtain the super-gauge invariant action

$$\mathcal{A} = \int_0^1 \left[-\frac{\dot{x}^2}{4\lambda} - \lambda V_p^2(x^1) - i\psi_n\dot{\psi}^n + 2i\lambda F_{mn}\psi^m\psi^n \right] d\tau, \tag{28}$$

which resembles to Berezin-Marinov action (Berezin and Marinov, 1975, 1977).

3. THE GREEN'S FUNCTION

Having shown how to formulate the problem of Dirac particle interacting with a pseudoscalar potential in the framework of Feynman-Beresin path integral, let us do integration over fermionic variables to express the Green's function only via bosonic path integrals. To begin, let us integrate over π and λ;

$$\begin{aligned} G^c = {} & \exp\left(i\gamma^n \frac{\partial_l}{\partial\theta^n}\right) \int_0^\infty d\lambda \int Dx \int Dp \int_{\psi(0)+\psi(1)=\theta} \mathcal{D}\psi \\ & \times \exp\left\{ i\int_0^1 d\tau \left[\lambda\left(p^2 - m^2 - V_p^2(x) + 2iF_{mn}\psi^m\psi^n\right) \right.\right. \\ & \left.\left. - i\psi_n\dot{\psi}^n + p\dot{x}\right] + \psi_n(1)\psi^n(0)\right\}\Big|_{\theta=0}. \end{aligned} \tag{29}$$

Then, we integrate over p_1, p_0 and x_0 ($x_0 \equiv t$) to obtain

$$G^c = \int \frac{dE}{2\pi} e^{iE(t_b - t_a)} G_E, \tag{30}$$

where the fixed energy Green's function G_E is given as follows

$$\begin{aligned} G_E = {} & \exp\left(i\gamma^n \frac{\partial_l}{\partial\theta^n}\right) \int_0^\infty d\lambda\, e^{i\lambda(E^2-m^2)} \int Dx^1 \\ & \times \exp\left\{ i\int_0^1 d\tau \left[\frac{(\dot{x}^1)^2}{4\lambda} - \lambda V_p^2(x^1)\right]\right\} \mathcal{I}(x^1,\lambda,\theta)\Big|_{\theta=0} \end{aligned} \tag{31}$$

and the factor $\mathcal{I}\left(x^1,\lambda,\theta\right)$ is given by

$$\begin{aligned} \mathcal{I}(x^1,\lambda,\theta) = {} & \int_{\psi(0)+\psi(1)=\theta} \mathcal{D}\psi \\ & \times \exp\left\{ \int_0^1 d\tau[\psi_n\dot{\psi}^n - 2\lambda F_{mn}\psi^m\psi^n] + \psi_n(1)\psi^n(0)\right\}. \end{aligned} \tag{32}$$

In order to calculate $\mathcal{I}\left(x^1,\lambda,\theta\right)$ we change, in the first stage, the integration variables from ψ to ξ, where

$$\psi = \frac{1}{2}\xi + \frac{\theta}{2}, \tag{33}$$

and the new variables ξ obey the following boundary condition

$$\xi(0)+\xi(1)=0. \tag{34}$$

Next, we change the proper time from τ to σ, where

$$d\sigma = V_p'(x^1)d\tau. \tag{35}$$

The factor $\mathcal{I}(x^1,\lambda,\theta)$ will be given then through the Grassmann Gaussian integral

$$\mathcal{I}(x^1,\lambda,\theta)=\exp\left(-\frac{\tilde{\lambda}}{2}f_{nm}\,\theta^n\theta^m\right)\int\mathcal{D}\xi \\ \times\exp\left\{\int_0^1\left[\frac{1}{4}\xi_n\dot{\xi}^n-\tilde{\lambda}f_{nm}\,\xi^n\xi^m-2\tilde{\lambda}f_{nm}\theta^n\xi^m\right]d\sigma\right\}\Bigg|_{\theta=0}, \tag{36}$$

where

$$\tilde{\lambda}=\lambda\int_0^1 V_p'(x^1)d\tau. \tag{37}$$

Since f_{nm} is constant, $\mathcal{I}(x^1,\lambda,\theta)$ has the same form as the spinor part of propagator corresponding to the problem of the constant electromagnetic field. It can be then evaluated to be

$$\mathcal{I}(x^1,\lambda,\theta)=\det{}^{\frac{1}{2}}(\cosh\tilde{\lambda}f)(1-B_{nm}\theta^n\theta^m), \tag{38}$$

where the tensor B is given by (see (Gitman *et al.* 1996, 1997))

$$B=\frac{1}{2}\tanh\left(\tilde{\lambda}f\right) \tag{39}$$

From the definition of the tensor f_{nm}, it is easy to show that

$$\cosh\left(\tilde{\lambda}f\right)=1+f^2\left(1-\cos\tilde{\lambda}\right) \tag{40}$$

and

$$B=\frac{1}{2}f\tan\tilde{\lambda}. \tag{41}$$

So, the fermionic part of the propagator is calculable

$$\exp\left(i\gamma^n\frac{\partial_l}{\partial\theta^n}\right)\mathcal{I}(x^1,\lambda,\theta)\Bigg|_{\theta=0}=\sum_{s=\pm1}\frac{1+s\gamma^0}{2}\exp\left(i\lambda s\int_0^1 V_p'(x^1)d\tau\right) \tag{42}$$

and the Green's function G_E can be expressed only through bosonic path integral over space coordinate x^1. It is given by

$$G_E=\sum_{s=\pm1}\frac{1+s\gamma^0}{2}\,P_s\left(x_b^1,x_a^1\right), \tag{43}$$

where

$$P_s(x_b^1, x_a^1) = \int_0^\infty d\lambda e^{i\lambda(E^2-m^2)}$$
$$\times \int Dx^1 \exp\left\{ i \int_0^\lambda d\tau \left[\frac{\left(\dot{x}^1\right)^2}{4} - \mathcal{U}_s(x^1) \right] \right\}, \quad (44)$$

and the effective potential $\mathcal{U}_s(x^1)$ has a supersymetric form

$$\mathcal{U}_s(x^1) = V_p^2(x^1) - sV_p'(x^1). \quad (45)$$

Thus, the problem of fermion interacting with a pseudoscalar potential becomes soluble in condition the effective potential is solvable in non relativistic case. Let us assume that $P_s(x_b^1, x_a^1)$ is integrable and has the following spectral decomposition

$$P_s\left(x_b^1, x_a^1\right) = \int_0^\infty d\lambda e^{i\lambda(E^2-m^2)} \sum_n e^{-i\lambda\mathcal{E}_n} \phi_n\left(x_b^1\right)\phi_n^*\left(x_a^1\right) \quad (46)$$

where $\mathcal{E}_n$ is the energy of non relativistic problem. Then by writing the matrix $\frac{1}{2}(1 + s\gamma^0)$ as a product of a spinor U and it's conjugate $\bar{U}$

$$\frac{1 + s\gamma^0}{2} = U_s\bar{U}_s, \quad (47)$$

with

$$U_{s=-1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad U_{s=+1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \quad (48)$$

and by integration over λ, we get

$$G_E = \sum_{n,s} U_s\bar{U}_s \frac{\phi_n\left(x_b^1\right)\phi_n^*\left(x_a^1\right)}{E^2 - E_n^2} \quad (49)$$

with

$$E_n^2 = m^2 + \mathcal{E}_n. \quad (50)$$

Integrating once again over E, we obtain

$$G^c(x_b, x_a) = \sum_{\epsilon=\pm1} \sum_{s=\pm1} \sum_{n=0}^{\infty} \Theta\left[\epsilon\left(t_b - t_a\right)\right] \varphi_{n,s}^\epsilon\left(x_b\right) \bar{\varphi}_{n,s}^\epsilon\left(x_a\right) \quad (51)$$

where $\Theta\left(\Delta t\right)$ is the Heaviside step function and the spinors $\varphi_{n,s}^\epsilon\left(x\right)$, which are given by

$$\varphi_{n,s}^\epsilon\left(x\right) = e^{-i\epsilon E_n t}\phi_n(x^1)U_s, \quad (52)$$

are solutions of the quadratic form of Dirac equation (due to the product K_-K_+). To obtain Dirac spinors solution of the problem in question we eliminate the superfluous states by acting the operator $K_+(x)$ on the spinors $\varphi^{\epsilon}_{n,s}(x)$

$$\psi_{n,s}(t,x) = N_{n,s}\left[i\gamma^{\mu}\frac{\partial}{\partial x^{\mu}} - \gamma^5 V_p(x^1) + m\right]\varphi^{\epsilon}_{n,s}(x), \tag{53}$$

where $N_{n,s}$ is a normalization constant. One can find

$$\psi_{n,s=+1}(x) = N_{n,+1}e^{-i\epsilon E_n t}\begin{pmatrix} E_n + m \\ i\frac{\partial}{\partial x^1} - iV_p(x^1)\end{pmatrix}\phi_n(x^1), \tag{54}$$

and

$$\psi_{n,s=-1}(x) = N_{n,-1}e^{-i\epsilon E_n t}\begin{pmatrix} -i\frac{\partial}{\partial x^1} - iV_p(x^1) \\ -E_n + m\end{pmatrix}\phi_n(x^1). \tag{55}$$

So, for any pseudoscalar potential, it is sufficient to calculate the path integral presented in (44) and to determine the energy spectrum E_n and the corresponding functions $\phi_n(x^1)$.

In the next section we give some explicit examples.

4. EXAMPLES

4.1. The Linear Potential (A Relativistic Oscillator)

Let us, first, consider the simpler case of the linear pseudoscalar potential

$$V_p(x^1) = m\omega x^1, \tag{56}$$

which describes a relativistic oscillator. The corresponding supersymetric potential has the same form as the usual harmonic oscillator plus a constant term

$$\mathcal{U}_s(x^1) = m^2\omega^2(x^1)^2 - sm\omega \tag{57}$$

and the relative kernel

$$P_s\left(x^1_b, x^1_a\right) = \int_0^{\infty} dT\, e^{iT\left(E^2 - m^2 + sm\omega\right)} \times \int Dx^1 \exp\left\{i\int_0^T d\tau\left[\frac{\left(\dot{x}^1\right)^2}{4} - m^2\omega^2(x^1)^2\right]\right\} \tag{58}$$

is then integrable. The functions $\phi_n(x^1)$ are those of the usual Harmonic oscillator

$$\phi_n(x^1) = \left(2^n n!\sqrt{\pi}\right)^{-\frac{1}{2}}(m\omega)^{\frac{1}{4}}\, e^{-\frac{m\omega}{2}(x^1)^2} H_n(\sqrt{m\omega}x^1) \tag{59}$$

and the energy spectrum is given by

$$E_n^2 = m^2 + (2n + 1 - s)\, m\omega. \tag{60}$$

4.2. The Modified Pöschl-Teller Potential

The second example that we give in this section is the potential

$$V_p(x^1) = V_0 \tanh x^1 - \frac{V_1}{\sinh 2x^1} \tag{61}$$

which leads to the modified Pöschl-Teller potential

$$\mathcal{U}_s(x^1) = V_0^2 + \frac{l^2 - \frac{1}{4}}{\sinh^2 x^1} - \frac{k^2 - \frac{1}{4}}{\cosh^2 x^1} \tag{62}$$

with

$$\begin{aligned} l &= \frac{V_1}{2} - \frac{s}{2} \\ k &= V_0 + \frac{V_1}{2} + \frac{s}{2}. \end{aligned} \tag{63}$$

The corresponding path integral

$$\begin{aligned} P_s\left(x_b^1, x_a^1\right) = &\int_0^\infty d\lambda e^{i\lambda\left(E^2 - m^2 - V_0^2\right)} \\ &\times \int Dx^1 \exp\left\{ i \int_0^\lambda d\tau \left[\frac{\left(\dot{x}^1\right)^2}{4} - \left(\frac{l^2 - \frac{1}{4}}{\sinh^2 x^1} - \frac{k^2 - \frac{1}{4}}{\cosh^2 x^1} \right) \right] \right\}, \end{aligned} \tag{64}$$

is also integrable. We have (Grosche, 1993)

$$\begin{aligned} \phi_n(x^1) = &N \left(\sinh x^1\right)^{k+1/2} \left(\cosh x^1\right)^{n-l+1/2} \\ &\times {}_2F_1\left(-n, k-n, l+1; \tanh^2 x^1\right), \end{aligned} \tag{65}$$

where

$$N = \frac{1}{\Gamma(k+1)} \left[\frac{2(l-k-2n-1)\Gamma(l-n)\Gamma(1+n+k)}{\Gamma(l-k-n)\, n!} \right]^{1/2}, \tag{66}$$

and

$$E_n^2 = m^2 + V_0^2 - \left(2n - V_0 + \frac{V_1}{2} + 1 - s \right)^2. \tag{67}$$

We remark here that we can obtain the Pöschl-Teller potential

$$\mathcal{U}_s(x^1) = \frac{l^2 - \frac{1}{4}}{\sin^2 x^1} - \frac{k^2 - \frac{1}{4}}{\cos^2 x^1} \tag{68}$$

from the pseudoscalar one

$$V_p(x^1) = V_0 \tan x^1 - \frac{V_1}{\sin 2x^1}, \tag{69}$$

however, we are not going to show details.

4.3. The Scarf II Potential

Another important example is the complex potential

$$V_p(x^1) = (p+q)\tanh x^1 - i\frac{(p-q)}{\cosh x^1}. \tag{70}$$

The supersymetric potential $\mathcal{U}_s(x^1)$ will be given by

$$\mathcal{U}_s(x^1) = \alpha - \beta + \beta \tanh^2 x^1 + \gamma \frac{\sinh x}{\cosh^2 x^1}, \tag{71}$$

where

$$\begin{aligned} \alpha &= (p+q)^2 \\ \beta &= -2(p^2+q^2) - s(p+q) \\ \gamma &= -i(p-q)[2(p+q)+s]. \end{aligned} \tag{72}$$

In this case, the relative Hamiltonian is pseudo-hermitian ($H^\dagger = \eta H \eta^{-1}$, where η may be the parity operator) and consequently the norm (pseudo-norm) is given, for scalar functions, by the integral (see Sinha and Roy, 2005 and references therein)

$$\int \psi(x)\,\psi^*(-x)\,dx. \tag{73}$$

Since the path integral representation for the Kernel $P_s\left(x_b^1, x_a^1\right)$ has a complex action it is convenient to use a Duru-Kleinert transformation (Kleinert, 1990), where the path x will be replaced by y, with

$$x^1 = h(y) \tag{74}$$

and the proper time λ by T, where

$$d\tau = f(x^1)d\sigma. \tag{75}$$

Note that the introduction of the function $f(x^1)$ in the Feynman propagator has brought the kinetic term to an inconvenient form containing a space dependant mass. By taking the following transformation

$$\sinh x^1 = -i \tanh y$$
$$\cosh x^1 = \frac{1}{\cosh y} \tag{76}$$

and by setting

$$f\left(x^1(y)\right) = \frac{-1}{\cosh^2 y} \tag{77}$$

the problem is solved. The discrete spectrum is given by the formula

$$E_{n,s}^2 = m^2 - \left(n + \frac{1-s}{2}\right)^2 - 2(p+q)\left(n + \frac{1-s}{2}\right) \tag{78}$$

and the pseudo-normalized $\phi_n(x^1)$ are given in terms of Jacobi Polynomials

$$\phi_n(x) = N\left(\frac{1 - i\sinh x^1}{2}\right)^{-\left(q+\frac{s-1}{4}\right)}\left(\frac{1 + i\sinh x^1}{2}\right)^{-\left(p+\frac{s-1}{4}\right)}$$
$$P_n^{\left(-2p-\frac{s}{2}, -2q-\frac{s}{2}\right)}\left(i\sinh x^1\right), \tag{79}$$

where

$$N = \sqrt{\frac{[2n - 2(p+q+s-1)]\,\Gamma\left(n+1-\left(p+q+\frac{s}{2}\right)\right) n!}{\Gamma\left(n+1-\left(2p+\frac{s}{2}\right)\right)\Gamma\left(n+1-\left(2q+\frac{s}{2}\right)\right)}}. \tag{80}$$

5. CONCLUSION

In this paper we have solved, by the path integral approach, the problem of Dirac particle interacting with a pseudoscalar potential in (1 + 1) dimension. The propagator of the particle is presented by means of supersymetric path integrals in the so called global projection, where the internal motion relative to the spin of the fermion is described by odd grassmannian variables. Since the pseudoclassical action has a more familiar form with respect to ψ-variables, we were able to express the Green's function only through bosonic path integrals. The problem has been reduced to a non relativistic one with a supersymetric potential.

Through the formulation given above and for the explicit examples analyzed in this paper, we conclude that the supersymetric path integrals are powerful method to study relativistic one fermion theory.

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